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THE MOTION OF A SHALLOW SEA UNDER INFLUENCE OF A NON-STATIONARY WIND-FIELD.

by

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The motion of a shallow sea under influence of a non-stationary wind-field [1]

H.A. Lauwerier

§1. Introduction.

This report is a contribution to the theory of the windeffect in the North Sea. We represent the North Sea by a rectangular basin of constant depth which borders to an ocean. The mathematical treatment is based upon the equations of motion for the total stream as given by Schalkwijk [2] viz.

$$\frac{\partial w_x}{\partial t} + \lambda w_x - \Omega w_y + gh \frac{\partial \zeta}{\partial x} = \frac{W_x}{\varrho} \qquad 1.1$$

$$\frac{\partial w_y}{\partial t} + \lambda w_y + \Omega w_x + gh \frac{\partial \zeta}{\partial y} = \frac{W_y}{\varrho} ,$$

to which we may add the equation of continuity

$$\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial \zeta}{\partial t} = 0. \qquad 1.2$$

At the boundary between the sea and the ocean, the depth of which is very large we assume the condition

$$\zeta = 0 , \qquad 1.3$$

whereas at the coast we have the condition that the component of the stream normal to the coast vanishes.

It has been found useful to consider not only the rectangular model but also simpler models such as an infinite strip or even a half-plane. These simpler models allow a mathematical treatment which is relatively simple and they may show phenomena of interest which also occur in the more complex model of the North Sea.

This report consists essentially of three parts.

In the first part it is shown that the Laplace transform of ζ satisfies a non-homogeneous Helmholtz equation with skew boundary conditions. Various applications of Green's theorem are discussed. The

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1) Research carried out under direction of Prof.Dr D. van Dantzig.

2) Schalkwijk, A contribution to the study of storm surges on the Dutch coast, 1947.

corresponding function of Green for a rectangle cannot be obtained by the usual analytic methods but if we solve a somewhat simpler problem of Green the solution of 1.1, 1.2, 1.3 can be reduced to a twodimensional nonsingular integral equation. This simpler problem of Green which corresponds to a stationary differential equation with non-stationary boundary conditions has been solved in the second part. In the case of a halfplane and a strip the exact problem of Green can be solved and the solution is discussed at length also in the first part.

In the third part the windeffect of a non-stationary but homogeneous wind upon a rotating halfplane sea is considered. Several numerical cases are studies in detail. The numerical constants correspond to those of the North Sea.

§2. The fundamental equations.

We consider a sea, the boundary of which consists of a coast C_1 and an ocean C_2. The arc coordinate of the boundary is indicated by s so that for increasing values of s the sea is on the left-hand side. The angle between the tangent and the x-axis is represented by τ which is a function of s. Differentiation along the boundary is indicated by $\frac{\partial}{\partial s}$ and differentiation along the normal pointing outwards by $\frac{\partial}{\partial n}$.

The fundamental equations of an arbitrary shallow-sea of variable depth h are

$$\left\{\begin{aligned} &\left(\frac{\partial}{\partial t}+\lambda\right) w_x - \Omega w_y + gh\frac{\partial\zeta}{\partial x} = \frac{W_x}{\varrho} \\ &\left(\frac{\partial}{\partial t}+\lambda\right) w_y + \Omega w_x + gh\frac{\partial\zeta}{\partial y} = \frac{W_y}{\varrho} \\ &\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial\zeta}{\partial t} = 0\,, \end{aligned}\right. \qquad 2.1$$

with the boundary conditions

$$\begin{aligned} &\text{at } C_1 \qquad w_x \sin\tau - w_y\cos\tau = 0\,, \\ &\text{at } C_2 \qquad \zeta = 0\,. \end{aligned} \qquad 2.2$$

Application of the Laplace transform

$$\bar{f}(p) \overset{\text{def}}{=} \int_{-\infty}^{\infty} e^{-pt} f(t)dt$$

$$f(t) = \int_{-i\infty}^{i\infty} e^{pt}\,\bar{f}(p)dp\,,$$

upon 2.1 gives

$$\left\{\begin{aligned} &(p+\lambda)\,\bar{w}_x - \Omega\bar{w}_y + gh\frac{\partial\bar{\zeta}}{\partial x} = \frac{\overline{W_x}}{\varrho} \\ &(p+\lambda)\,\bar{w}_y + \Omega\bar{w}_x + gh\frac{\partial\bar{\zeta}}{\partial y} = \frac{\overline{W_y}}{\varrho} \end{aligned}\right. \qquad 2.3$$

$$\frac{\partial\bar{w}_x}{\partial x} + \frac{\partial\bar{w}_y}{\partial y} + p\,\bar{\zeta} = 0\,, \qquad 2.4$$

with the boundary conditions

$$\text{at } C_1 \qquad \bar{w}_x\sin\tau - \bar{w}_y\cos\tau = 0\,, \qquad 2.5$$

$$\text{at } C_2 \qquad \bar{\zeta} = 0\,. \qquad 2.6$$

If we solve $\bar{w}_x$ and $\bar{w}_y$ from the two equations 2.3 we find

$$\{(p+\lambda)^2+\Omega^2\} \quad \bar{w}_x = -gh\left\{(p+\lambda)\frac{\partial\bar{\zeta}}{\partial x}+\Omega\frac{\partial\bar{\zeta}}{\partial y}\right\}+\frac{1}{\rho}\left\{(p+\lambda)\bar{W}_x+\Omega\bar{W}_y\right\},$$

2.7

$$\{(p+\lambda)^2+\Omega^2\} \quad \bar{w}_y = -gh\left\{(p+\lambda)\frac{\partial\bar{\zeta}}{\partial y}-\Omega\frac{\partial\bar{\zeta}}{\partial x}\right\}+\frac{1}{\rho}\left\{(p+\lambda)\bar{W}_y-\Omega\bar{W}_x\right\}.$$

Substitution into 2.4 gives a partial differential equation for $\bar{\zeta}$ only

$$\Delta\bar{\zeta}+A\frac{\partial\bar{\zeta}}{\partial x}+B\frac{\partial\bar{\zeta}}{\partial y}-k^2\bar{\zeta}=\bar{F}, \qquad 2.8$$

where

$$A=\frac{\partial \ln h}{\partial x}-\frac{\Omega}{p+\lambda}\frac{\partial \ln h}{\partial y},$$

$$B=\frac{\partial \ln h}{\partial y}+\frac{\Omega}{p+\lambda}\frac{\partial \ln h}{\partial x},$$

$$k^2=\frac{p\{(p+\lambda)^2+\Omega^2\}}{gh\,(p+\lambda)}, \qquad 2.9$$

$$\bar{F}=\frac{1}{gh\rho}\left\{\left(\frac{\partial\bar{W}_x}{\partial x}+\frac{\partial\bar{W}_y}{\partial y}\right)+\frac{\Omega}{p+\lambda}\left(\frac{\partial\bar{W}_y}{\partial x}-\frac{\partial\bar{W}_x}{\partial y}\right)\right\}. \qquad 2.10$$

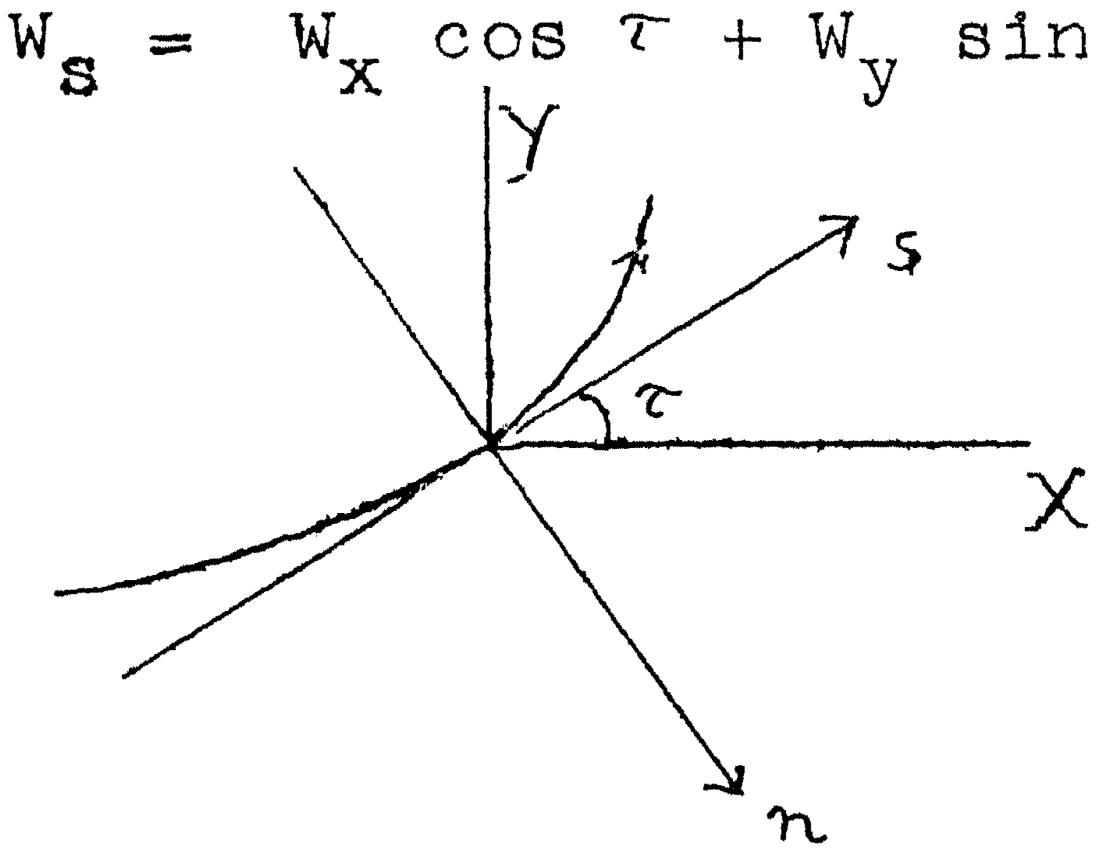
The boundary conditions become with respect to $\bar{\zeta}$

at C_1 $$(p+\lambda)\frac{\partial\bar{\zeta}}{\partial n}+\Omega\frac{\partial\bar{\zeta}}{\partial s}=\frac{1}{gh\rho}\left\{(p+\lambda)\bar{W}_n+\Omega\bar{W}_s\right\}, \qquad 2.11$$

at C_2 $$\bar{\zeta}=0. \qquad 2.12$$

In formula 2.11 W_n and W_s represent the normal and the tangential component of the windvector

$$W_n = W_x \sin\tau - W_y\cos\tau,$$
$$W_s = W_x\cos\tau + W_y\sin\tau.$$



In this report we consider only the case of a sea with a constant depth. The differential equation 2.8 of becomes simpler since A=B=O

$$\Delta\bar{\zeta}-k^2\bar{\zeta}=\bar{F}. \qquad 2.13$$

The meaning of k^2 and $\overline{F}$, and the boundary conditions remain the same.

For the rectangular model of the North Sea we consider the rectangle $|x| < a \quad 0 < y < b$ where the side $y=b$ represents the ocean boundary. The Dutch coast corresponds to the x-axis so that the origin might be considered as representing Den Helder. The leak of the Channel and of the Kattegat which is less important are neglected in this model.

The corresponding boundary conditions with respect to $\overline{\zeta}$ are

$$x = \pm a \qquad (p+\lambda)\frac{\partial\overline{\zeta}}{\partial x} + \Omega\frac{\partial\overline{\zeta}}{\partial y} = \frac{1}{gh\rho}\left\{(p+\lambda)\overline{W}_x + \Omega\overline{W}_y\right\}, \qquad 2.14$$

$$y = 0 \qquad (p+\lambda)\frac{\partial\overline{\zeta}}{\partial y} - \Omega\frac{\partial\overline{\zeta}}{\partial x} = \frac{1}{gh\rho}\left\{(p+\lambda)\overline{W}_y - \Omega\overline{W}_x\right\}, \qquad 2.15$$

$$y = b \qquad \overline{\zeta} = 0 \qquad 2.16$$

§3. Green's theorem.

Consider a closed and simply connected region R with boundary C and two functions $\overline{G}(x,y,\xi,\eta,p)$, $\overline{H}(x,y,p)$ with the following properties

a $\overline{G}$ has a logarithmic singularity at (ξ,η) of the following kind

$$\overline{G} \sim -\frac{1}{2\pi}\ln r + O(1), \qquad 3.1$$

where

$$r \overset{\text{def}}{=} \left\{(x-\xi)^2 + (y-\eta)^2\right\}^{\frac{1}{2}},$$

and has everywhere else in R continuous first and second derivatives.

b $\overline{H}$ has continuous first and second derivatives in R.

Green's theorem states that

$$\overline{H}(\xi,\eta) = \iint_R \left\{\overline{G}(x,y,\xi,\eta)\,\Delta\overline{H}(x,y) - \overline{H}(x,y)\,\Delta\overline{G}(x,y,\xi,\eta)\right\} dx\,dy -$$
$$- \int_C \left\{\overline{G}\frac{\partial\overline{H}}{\partial n} - \overline{H}\frac{\partial\overline{G}}{\partial n}\right\} ds, \qquad 3.2$$

We take $\overline{H} = \overline{\zeta}$ satisfying 2.13, 2.11 along the coast C_1 and 2.12 along the ocean C_2. Then the first part of the last term in the right-hand side of 3.2 becomes

$$\int_C \overline{G}\frac{\partial\overline{\zeta}}{\partial n} ds = \int_{C_1} \overline{G}\frac{\partial\overline{\zeta}}{\partial n} ds + \int_{C_2} \overline{G}\frac{\partial\overline{\zeta}}{\partial n} ds,$$

$$\int_{C_1} \overline{G}\frac{\partial\overline{\zeta}}{\partial n} ds = \int_{C_1} \overline{G}\,\overline{\Gamma}\, ds - \frac{\Omega}{p+\lambda}\int_{C_1} \overline{G}\frac{\partial\overline{\zeta}}{\partial s} ds,$$

with

$$\bar{F} \overset{def}{=} \frac{1}{gh\rho} \left\{ \bar{W}_n + \frac{\Omega}{p+\lambda} \bar{W}_s \right\} . \qquad 3.3$$

The second term can be integrated by parts

$$\int_{C_1} \bar{G} \frac{\partial \bar{\zeta}}{\partial s} ds = - \int_{C_1} \bar{\zeta} \frac{\partial \bar{G}}{\partial s} ds,$$

as the integrated part vanishes because of 2.12 and of the continuity of $\bar{\zeta}$. Then 3.2 passes into

$$\bar{\zeta}(\xi,\eta,p) = - \iint_R \zeta(x,y,p)(\Delta - k^2)\bar{G}(x,y,\xi,\eta,p) \, dx \, dy -$$

$$- \int_{C_2} \bar{G}(s,\xi,\eta,p) \frac{\partial \bar{\zeta}(s,p)}{\partial n} ds + \qquad 3.4$$

$$+ \int_{C_1} \bar{\zeta}(s,p) \left\{ \frac{\partial \bar{G}(s,\xi,\eta,p)}{\partial n} - \frac{\Omega}{p+\lambda} \frac{\partial \bar{G}(s,\xi,\eta,p)}{\partial s} \right\} ds + \bar{A}(\xi,\eta,p),$$

with

$$\bar{A}(\xi,\eta,p) = \iint_R \bar{G}(x,y,\xi,\eta,p) \, \bar{F}(x,y,p) dx \, dy - \int_{C_1} \bar{G}(s,\xi,\eta,p) \, \bar{F}(s,p) ds. \qquad 3.5$$

for any known function $\bar{G}$ and any given windfield.

With regard to the other term several possibilities arise.

1. If we solve Green's problem i.e. if a solution can be obtained fr. the differential equation

$$\Delta \bar{G} = k^2 \bar{G} , \qquad 3.6$$

which has the logarithmic singularity 3.1 in (ξ,η) and which satisfies the boundary conditions

$$\begin{cases} (p+\lambda) \frac{\partial \bar{G}}{\partial n} - \Omega \frac{\partial \bar{G}}{\partial s} = 0 & \text{along } C_1 , \\ \bar{G} = 0 & \text{along } C_2 , \end{cases} \qquad 3.7$$

from 3.4 an explicit solution for $\bar{\zeta}$ is obtained viz.

$$\bar{\zeta}(\xi,\eta,p) = \iint_R \bar{G}(x,y,\xi,\eta,p) \, \bar{F}(x,y,p) dx \, dy - \int_{C_1} \bar{G}(s,\xi,\eta,p) \, \bar{F}(s,p) ds \qquad 3.8$$

If the originals of the Laplace transform $\overline{F}$, $\overline{f}$ and $\overline{G}(x,y,\xi,\eta,p)$ are denoted by F, f, $G(x,y,\xi,\eta,t)$ respectively, the original $\zeta(\xi,\eta,t)$ can be written as follows

$$\zeta(\xi,\eta,t)=\iint_R dx\,dy \int_{-\infty}^{\infty} G(x,y,\xi,\eta,t-\theta)\,F(x,y,\theta)\,d\theta\ -$$

$$-\int_{C_1} ds \int_{-\infty}^{\infty} G(s,\xi,\eta,t-\theta)\,f(s,\theta)\,d\theta\,. \qquad 3.9$$

From 2.10 and 3.3 we have moreover

$$F(x,y,t)=\frac{1}{gh\rho}\left(\frac{\partial W_x}{\partial x}+\frac{\partial W_y}{\partial y}\right)+\frac{\Omega}{gh\rho}e^{-\lambda t}\int_{-\infty}^{t} e^{\lambda\theta}\left(\frac{\partial W_y}{\partial x}-\frac{\partial W_x}{\partial y}\right)d\theta\,, \qquad 3.10$$

and

$$f(s,t)=\frac{1}{gh\rho}W_n+\frac{\Omega}{gh\rho}e^{-\lambda t}\int_{-\infty}^{t} e^{\lambda\theta}\,W_s\,d\theta\,. \qquad 3.11$$

However the direct solution of the Green problem becomes very complicated and only for seas of a very special form, apart from the trivial case of the whole plane we mention a halfplane, a strip and a quadrant, a solution could be obtained.

2. We restrict ourselves to the discussion of the rectangular case which is most interesting to us.

a. As indicated in a report by Braakman (Febr.1954) we can easily obtain a solution of the differential equation 2.13 having the right singularity at (ξ,η) and which satisfies the boundary condition at the ocean frontier without bothering about the boundary condition at the coast. Then 3.4 gives us a singular integral equation for $\overline{\zeta}$ along the coastline, which can be solved by the method of H. Poincaré and developed in particular by the Tiflis-school [1].
This equation is of the following form

$$\overline{\zeta}=\int_{C_1}\overline{\zeta}(s)\left(\frac{\partial\overline{G}}{\partial n}-\operatorname{cotg}\omega\,\frac{\partial\overline{G}}{\partial s}\right)ds+\overline{A}\quad. \qquad 3.12$$

b. It is possible to obtain a solution of 2.13 which satisfies not only the boundary condition at the ocean frontier but also the boundary

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1) N. Muskhelishvili, Singular integral equations. Noordhoff 1954.

condition at the south coast. Then we are left with a singular integral equation for $\bar{\zeta}$ along the West- and East coast only.

c. As an alternative to the preceding method we can also obtain a solution of 2.13 which satisfies the boundary conditions at the West- and East coast. In this case a singular integral equation for $\bar{\zeta}$ along the ocean and the South coast results.

Evidently by these methods the difficulties are displaced towards the solution of the singular integral equation.

3. If we take a function $\bar{G}$ having the right singularity which satisfies all boundary conditions without bothering about the differential equation we are left with a regular integral equation over the sea

$$\bar{\zeta}(\xi,\eta) = - \iint_R \bar{\zeta}(x,y)(\Delta - k^2)\, \bar{G}(x,y,\xi,\eta)\,dx\,dy + \bar{A}(\xi,\eta) \; . \qquad 3.13$$

For $\bar{G}$ e.g. a harmonic function can be taken i.e.

$$\Delta \bar{G} = 0 \; ,$$

in R, but evidently this has no influence of importance upon the type of the integral equation to be solved subsequently. However, for k=0 the integral equation passes into an explicit representation of $\bar{\zeta}$. Since k=0 corresponds to the stationary case which is reached for $t \to \infty$ this method has advantages if a solution is sought which does not deviate strongly from the stationary solution. Furthermore this method has the advantage that use can be made of the properties of analytic functions.

We conclude this section by giving an outline of the programme to be followed in the following sections.

In order to gain some insight into the nature of the solution of the rectangular case it will be found useful to consider first the cases mentioned under method 1 whilst passing from the simplest cases, plane and halfplane, to more complex cases.

Having obtained the Laplace transform of a solution we are left with the back transformation. In the general case of a finite sea the poles of the Laplace transform $\bar{\zeta}$ of $\zeta(x,y,t)$ correspond to the periods of the proper oscillations of the sea. If $\bar{\zeta}$ is developed according to its poles the expansion of ζ in eigenfunctions is obtained. If $\bar{\zeta}$ is developed into a series for small p an expansion of ζ is obtained which is useful for large time values.

Using a method, originally applied by Taylor to the case of the Irish Sea, Van Dantzig and Veltkamp[1] have solved the eigenvalue problem. It appears that the periods of the proper oscillations can be determined from an infinite determinant. Any knowledge about the periods of the proper oscillations is extremely useful since it allows a considerable improvement of the usefulness of the expansions of ζ .

§4. Plane.

The following problem will be considered
To determine a function $w(x,y,\xi,\eta)$ which satisfies

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = w , \qquad 4.1$$

$$w \backsim -\ln r \qquad r = \sqrt{(x-\xi)^2+(y-\eta)^2} \text{ in } (\xi,\eta), \qquad 4.2$$

$$w \to 0 \quad \text{for} \quad x^2 + y^2 \to \infty . \qquad 4.3$$

The Helmholtz equation 4.1 has the elementary solution

$$e^{\alpha x+\beta y} \quad \text{with} \quad \alpha^2+\beta^2=1 ,$$

and if $f(x,y)$ is a solution also $f(x+a, y+b)$ is a solution.

From the superposition principle it is clear that

$$w= \frac{1}{2} \int_{-\infty+\gamma i}^{\infty+\gamma i} \varphi(v) \; e^{-x\operatorname{ch}v-yi\operatorname{sh}v} \, dv \qquad 4.4$$

also satisfies 4.1.

From the property $\int_z^{\infty} e^{-t} \frac{dt}{t} \backsim -\ln z$ for $z \to 0$,

it follows that, if for $v \to \pm\infty$ $\varphi(v) \to 1$, the integral 4.4 behaves as $-\frac{1}{2}\ln(x+yi) - \frac{1}{2}\ln(x-yi) = -\ln\sqrt{x^2+y^2}$.
The solution defined by 4.4 is valid only in the halfplane

$$x \cos\gamma - y \sin\gamma > 0 ,$$

but by varying γ the solution can be continued through the whole plane.

In the case $\varphi(v)=1$ 4.4 has only the singularity at the origin and the solution thus obtained clearly satisfies the condition 4.2 in the origin and the condition 4.3. Thus we have found the following solution of 4.1, 4.2 and 4.3

$$w = \frac{1}{2} \int_{-\infty+\gamma i}^{\infty+\gamma i} \exp\{ -(x-\xi)\operatorname{ch}v - (y-\eta)i\operatorname{sh}v \} \, dv , \qquad 4.5$$

1) Prof.Dr D. van Dantzig and G.W. Veltkamp, On the free oscillations of a rotating rectangular sea. Report T.W.32 (1955).

or

$$w = \tfrac{1}{2} \int_{-\infty}^{\infty} \exp \left\{ - |x-\xi| \operatorname{ch} v - (y-\eta) i \operatorname{sh} v \right\} dv. \qquad 4.6$$

If in 4.5 the following substitution is performed

$$\left\{ (x-\xi) + i(y-\eta) \right\} e^{v} = t$$

we obtain

$$w = \tfrac{1}{2} \int_{0}^{\infty} \exp - \left(\frac{t}{2} + \frac{r^2}{2t}\right) \frac{dt}{t} . \qquad 4.7$$

From the theory of the Bessel functions it follows that

$$w = K_0(r) , \qquad 4.8$$

where $K_0(r)$ is a Bessel function of the third kind which vanishes for $r \to \infty$.

The Green function of the Helmholtz equation for the full plane becomes according to the notation of § 3

$$\bar{G}(x,y,\xi,\eta) = \frac{1}{2\pi} K_0(kr) , \qquad 4.9$$

with

$$k^2 = \frac{p \left\{ (p+\lambda)^2 + \Omega^2 \right\}}{p+\lambda} .$$

§ 5. Halfplane.

The following problem will be considered

To determine a function $w(x,y,\xi,\eta)$ which satisfies

$$y > 0 \qquad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = w , \qquad 5.1$$

$$w \sim - \ln r \qquad r = \sqrt{(x-\xi)^2 + (y-\eta)^2} \text{ in } (\xi,\eta) , \qquad 5.2$$

$$w \to 0 \quad \text{at infinity} , \qquad 5.3$$

and with the following boundary condition

$$\frac{\partial w}{\partial x} \cos \omega + \frac{\partial w}{\partial y} \sin \omega = 0. \qquad 5.4$$

The left-hand side of 5.4 represents differentiation along a line ℓ of direction ω

$$\frac{\partial \varphi(x,y)}{\partial \ell} \equiv \frac{d}{d\ell} \varphi \left\{ x_0 + \ell \cos \omega , y_0 + \ell \sin \omega \right\} ,$$

and the condition 5.4 is often called a skew boundary condition.

We shall try to determine the solution of this problem as the sum of two functions the first of which has the appropriate behaviour at (ξ,η) and infinity and the second of which is regular in the halfplane $y > 0$ and also satisfies 5.3. We write

$$w = w_1 + w_2 ,$$

where

$$w_1 = K_0(r) = \tfrac{1}{2} \int_{-\infty}^{\infty} \exp\left\{ - |y-\eta| \operatorname{ch}v - (x-\xi) i \operatorname{sh}v \right\} dv ,$$

and

$$w_2 = \tfrac{1}{2} \int_{-\infty}^{\infty} \varphi(v) \exp\left\{ - |y+\eta| \operatorname{ch}v - (x-\xi) i \operatorname{sh}v \right\} dv .$$

The latter function, which is singular at the reflected point $(\xi,-\eta)$ contains an arbitrary function $\varphi(v)$ which will be determined in such a way that w_1+w_2 satisfies the boundary condition 5.4.

Substitution of w_1 and w_2 into 5.4 leads easily to

$$\varphi(v) = \frac{\operatorname{ch}v \sin\omega - i \operatorname{sh}v \cos\omega}{\operatorname{ch}v \sin\omega + i \operatorname{sh}v \cos\omega} = - \frac{\operatorname{sh}(v+i\omega)}{\operatorname{sh}(v-i\omega)} . \qquad 5.5$$

Thus we have

$$w = K_0(r) - \tfrac{1}{2} \int_{-\infty}^{\infty} \frac{\operatorname{sh}(v+i\omega)}{\operatorname{sh}(v-i\omega)} \exp\left\{ -(y+\eta)\operatorname{ch}v - (x-\xi) i \operatorname{sh}v \right\} dv, \qquad 5.6$$

which is valid for $y \geqslant 0$.

The solution 5.6 can be continued into the lower halfplane across the boundary $y = 0$. It is sufficient to consider only the continuation of the integral

$$I(r,\theta) \overset{\text{def}}{=} \tfrac{1}{2} \int_{-\infty}^{\infty} \frac{\operatorname{sh}(v+i\omega)}{\operatorname{sh}(v-i\omega)} \exp -\left\{ r \sin(\theta + vi) \right\} dv, \qquad 5.7$$

where $y+\eta = r \sin\theta$, $x-\xi = r\cos\theta$, and $0<\theta<\pi$.

Of course this has nothing to do with the analytic continuation of complex functions since $\frac{\partial^2 I}{\partial x^2} + \frac{\partial^2 I}{\partial y^2} = I \neq 0$. However in a similar way as there we may obtain a continuation of 5.7 for $-\pi < \theta < 0$.

If we take the horizontal path of integration

$$\operatorname{Im} v = -\alpha$$

5.7 is valid in the interval $-\alpha < \theta < \pi-\alpha$

However, this is only possible if no pole of $\frac{\operatorname{sh}(v+i\omega)}{\operatorname{sh}(v-i\omega)}$ is passed when the path of integration is moved from $\operatorname{Im} v = 0$ towards $\operatorname{Im} v = -\alpha$.

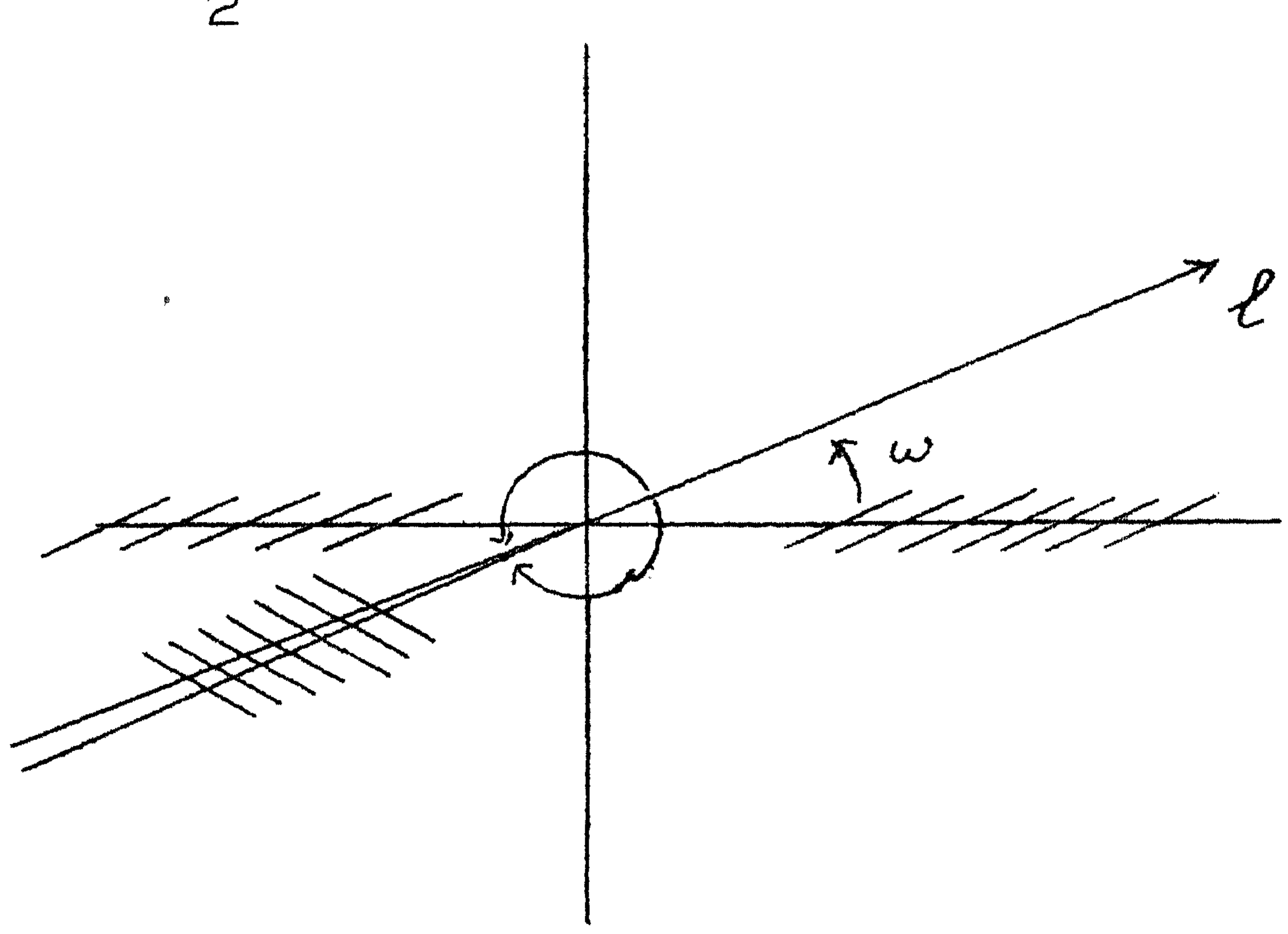
This function has a series of poles

$$v = \omega i + n \pi i \qquad 5.8$$

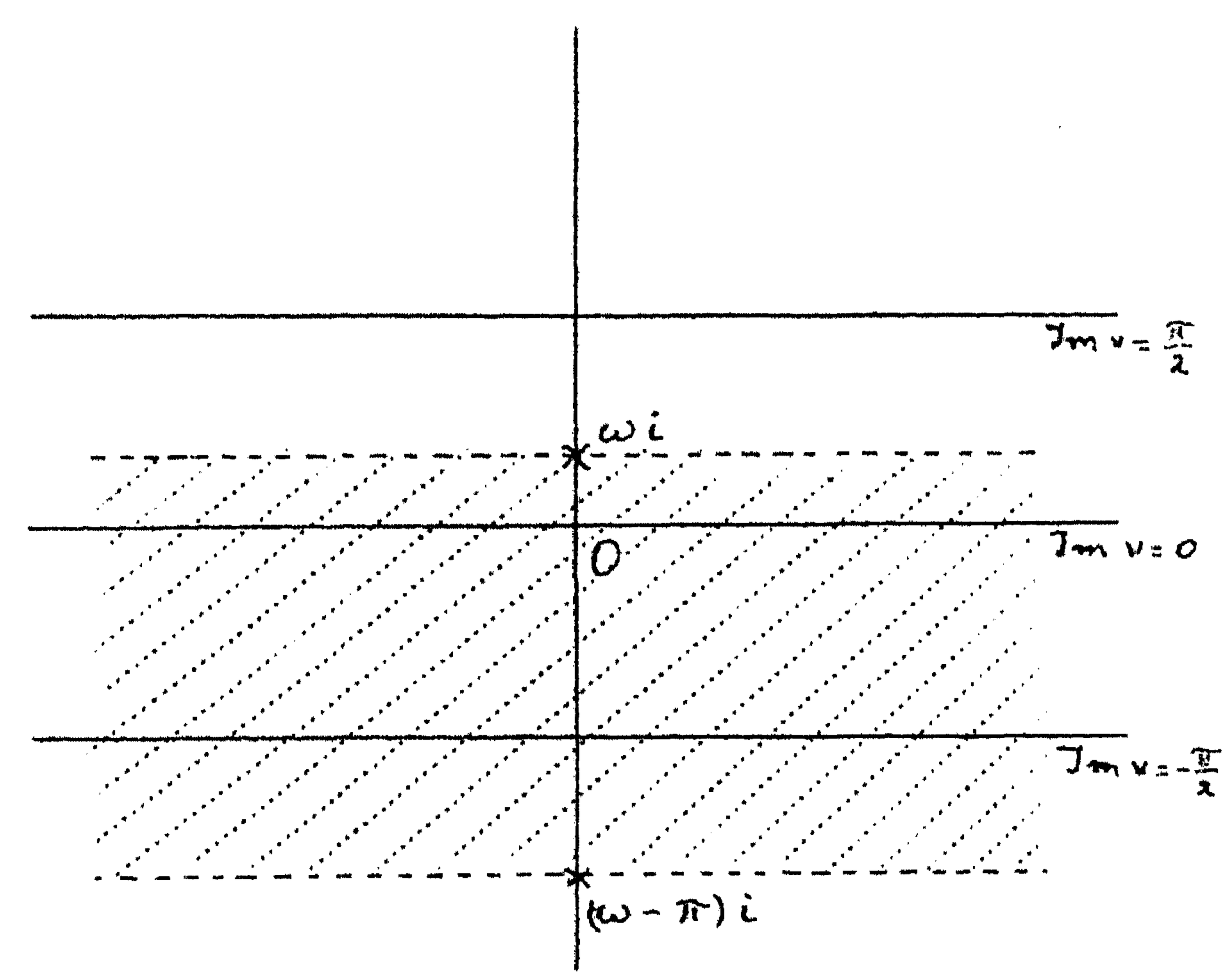
so that there is a free region of width π in which the path of integration can be moved without passing poles. This means that the remaining halfplane $y < 0$ i.e. $-\pi < \theta < 0$ 0 can be covered completey with the exception only of one critical halfline. It is obvious that the function $I(r,\theta)$ obtained in this way is regular in the slit plane $\theta \neq \theta_c$ and that at the halfline $\theta = \theta_c$ $I(r,\theta)$ is discontinuous and makes a jump. The magnitude of the jump is determined by the corresponding pole viz.

$$2\pi \sin 2\omega . \qquad 5.9$$

The behaviour of 5.7 is illustrated below for a real ω with $0 < \omega < \frac{\pi}{2}$.



r, θ - plane



v - plane

If in 5.7 new variables are introduced by means of

$$\bar{x} = x - \xi \qquad p = \bar{x}\sin\omega - \bar{y}\cos\omega$$
$$\qquad\qquad\qquad 5.10$$
$$\bar{y} = y + \eta \qquad q = \bar{x}\sin\omega + \bar{y}\cos\omega$$

we may write

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\mathrm{sh}(v+i\omega)}{\mathrm{sh}(v-i\omega)} \exp\left\{ - \frac{p}{\sin 2\omega}\,\mathrm{sh}(v+i\omega) + \frac{q}{\sin 2\omega}\,\mathrm{sh}(v-i\omega) \right\} dv.$$

This in turn can be written as

$$I = \left\{ \frac{\partial}{\partial p} \int_q^{\infty} dq' \right\} \frac{1}{2} \int_{-\infty}^{\infty} \exp\left\{ - \frac{p}{\sin 2\omega}\,\mathrm{sh}(v+i\omega) - \frac{q'}{\sin 2\omega}\,\mathrm{sh}(v-i\omega) \right\} dv,$$

or

$$I = \frac{\partial}{\partial p} \int_q^{\infty} K_0 \left\{ \sqrt{\left(\frac{q'-p}{2\cos\omega}\right)^2 + \left(\frac{q'+p}{2\sin\omega}\right)^2} \right\} dq' ,$$ and finally

$$I = \frac{\partial}{\partial p} \int_0^\infty K_o \left\{ \sqrt{(\bar{x} + \frac{\lambda}{2\sin\omega})^2 + (\bar{y} + \frac{\lambda}{2\cos\omega})^2} \right\} d\lambda , \qquad 5.11$$

where $\frac{\partial}{\partial p} = \frac{\partial}{\partial(\bar{x}\sin\omega - \bar{y}\cos\omega)}$ i.e. differentiation symmetrical to ℓ with respect to y = 0, is to be interpreted as a differentiation with $q = \bar{x}\cos\omega + \bar{y}\sin\omega$ being constant.

The integral over K_o can be interpreted as an assembly of contributions from logarithmic singularities $(-\frac{\lambda}{2\sin\omega}, -\frac{\lambda}{2\cos\omega})$ lying on the halfline of discontinuities $p = \bar{x}\sin\omega - \bar{y}\cos\omega = 0$.

The differentiation $\frac{\partial}{\partial p}$ for constant q can be interpreted by saying that the logarithmic singularities are converted into dipoles the vector of each of which is symmetrical to the direction ℓ of differentiation.

Of course any dipole may be decomposed into two components, one along the tail and one normal to it. Then the dipoles in the direction normal to the tail get the density $\sin 2\omega$. The components along the tail destroy each other except at the end of the tail $(\xi,-\eta)$ where a single pole of strength $\cos 2\omega$ results.

It is of interest to study the behaviour of the solution 5.6 near the dipole line somewhat more in detail.

If we want to know what happens at a line of dipoles we may start from the solution

$$w_o = \int_0^\infty K_o \left(\sqrt{(x-\xi)^2 + y^2}\right) d\xi , \qquad 5.12$$

which has a uniform distribution of simple poles on the positive part of the x-axis.

If for K_o the following expression is substituted

$$K_o \left(\sqrt{(x-\xi)^2+y^2}\right) = \tfrac{1}{2} \int_{-\infty}^{\infty} e^{-|x-\xi|\mathrm{ch}s - yi\,\mathrm{sh}s} ds ,$$

and if the integrations are interchanged we find

$$w_o = \tfrac{1}{2} \int_{-\infty}^{\infty} e^{-yi\,\mathrm{sh}s} ds \int_0^\infty e^{-|x-\xi|\mathrm{ch}s} d\xi .$$

The last integration can be carried out

$$\int_0^\infty e^{-|x-\xi|\mathrm{ch}s} d\xi = \begin{cases} \dfrac{e^{x\mathrm{ch}s}}{\mathrm{ch}s} & \text{if } x < 0 \\ \dfrac{2-e^{-x\mathrm{ch}s}}{\mathrm{ch}s} & \text{if } x > 0 , \end{cases}$$

so that

$$w_o = \frac{1}{2} \int_{-\infty}^{\infty} \left\{ 1+\frac{x}{|x|} - \frac{x}{|x|} e^{-|x| chs} \right\} \frac{e^{-yishs}}{chs} ds$$

$$= \frac{1}{2} \left(1+\frac{x}{|x|}\right) \int_{-\infty}^{\infty} \frac{e^{-yip}}{p^2+1} dp - \frac{1}{2}\frac{x}{|x|} \int_{-\infty}^{\infty} \frac{e^{-|x| chs-yishs}}{chs} ds ,$$

and finally

$$w_o = \frac{\pi}{2} \left(1+\frac{x}{|x|}\right) e^{-|y|} - \frac{1}{2}\frac{x}{|x|} \frac{e^{-|x| chs-yishs}}{chs} ds . \qquad 5.13$$

Obviously w_o is only singular at $y=0$ $x > 0$. The singular behaviour is determined by the first term in the right-hand side of 5.13 which becomes for $x > 0$

$$\pi e^{-|y|} ,$$

therefore w_o and all even derivatives with respect to y are continuous whereas all odd derivatives with respect to y are discontinuous and make jumps of magnitude 2π.

From 5.12 the solution with a uniform distribution of dipoles can be derived by differentiation.

If the dipoles on the positive x-axis are orientated along the x-axis the solution is $-\frac{\partial w_o}{\partial x}$ or from 5.12

$$-\frac{\partial w_o}{\partial x} = - K_o \left(\sqrt{x^2+y^2}\right) , \qquad 5.14$$

so that the horizontal dipoles reduce to a single pole at the origin.

If the dipoles are orientated normal to the x-axis we have the solution $-\frac{\partial w_o}{\partial y}$, thus for $x > 0$

$$-\frac{\partial w_o}{\partial y} = -\pi \frac{y}{|y|} e^{-|y|} + \frac{i}{2} \int_{-\infty}^{\infty} ths\, e^{-|x| chs - yishs} ds, \qquad 5.15$$

so that in this case the function and all even derivations with respect to y are discontinuous with jumps 2π whereas all odd derivatives with respect to y are continuous.

If the dipoles are orientated in an arbitrary direction θ the solution is $-\frac{\partial w_o}{\partial x}\cos\theta - \frac{\partial w_o}{\partial y}\sin\theta$ with a pole of strength $-\cos\theta$ at the origin and discontinuities at $y=0$ $x>0$ determined by $\frac{y}{|y|}\sin\theta\ e^{-|y|}$.

§6. A strip.

To determine a function $w(x,y,\xi,\eta)$ which satisfies

$$-a < x < a \qquad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = w, \tag{6.1}$$

$$x = \pm a \quad \frac{\partial w}{\partial x} \sin\omega - \frac{\partial w}{\partial y}\cos\omega = 0, \tag{6.2}$$

$$w \sim -\ln r \quad \text{in } (\xi, \eta).$$

We try to find the solution in the following form

$$w = K_0(r) + \tfrac{1}{2}\int_{-\infty}^{\infty} \left\{ \varphi_1 e^{x\,\mathrm{ch}v} + \varphi_2 e^{-x\,\mathrm{ch}v} \right\} e^{-(y-\eta)i\,\mathrm{sh}v}\, dv. \tag{6.3}$$

It is known that

$$K_0(r) = \tfrac{1}{2}\int_{-\infty}^{\infty} \exp\left\{ -|x-\xi|\,\mathrm{ch}v - (y-\eta)i\,\mathrm{sh}v \right\} dv.$$

Therefore 6.2 gives

$$x=-a \qquad \varphi_1 e^{-a\,\mathrm{ch}v}\,\mathrm{sh}(v-\omega i) - \varphi_2 e^{a\,\mathrm{ch}v}\,\mathrm{sh}(v+\omega i) + \mathrm{sh}(v-\omega i)\exp\left\{-(a+\xi)\mathrm{ch}v\right\} = 0,$$

$$x=a \qquad \varphi_1 e^{a\,\mathrm{ch}v}\,\mathrm{sh}(v-\omega i) - \varphi_2 e^{-a\,\mathrm{ch}v}\,\mathrm{sh}(v+\omega i) - \mathrm{sh}(v+\omega i)\exp\left\{-(a-\xi)\mathrm{ch}v\right\} = 0,$$

from which φ_1 and φ_2 can be solved

$$2\,\mathrm{sh}(2a\,\mathrm{ch}v).\ \varphi_1 = e^{-(2a+\xi)\mathrm{ch}v} + \frac{\mathrm{sh}(v+\omega i)}{\mathrm{sh}(v-\omega i)} e^{\xi\,\mathrm{ch}v}, \tag{6.4}$$

$$2\,\mathrm{sh}(2a\,\mathrm{ch}v).\ \varphi_2 = e^{-(2a-\xi)\mathrm{ch}v} + \frac{\mathrm{sh}(v-\omega i)}{\mathrm{sh}(v+\omega i)} e^{-\xi\,\mathrm{ch}v}.$$

Substitution of 6.4 into 6.3 gives the result

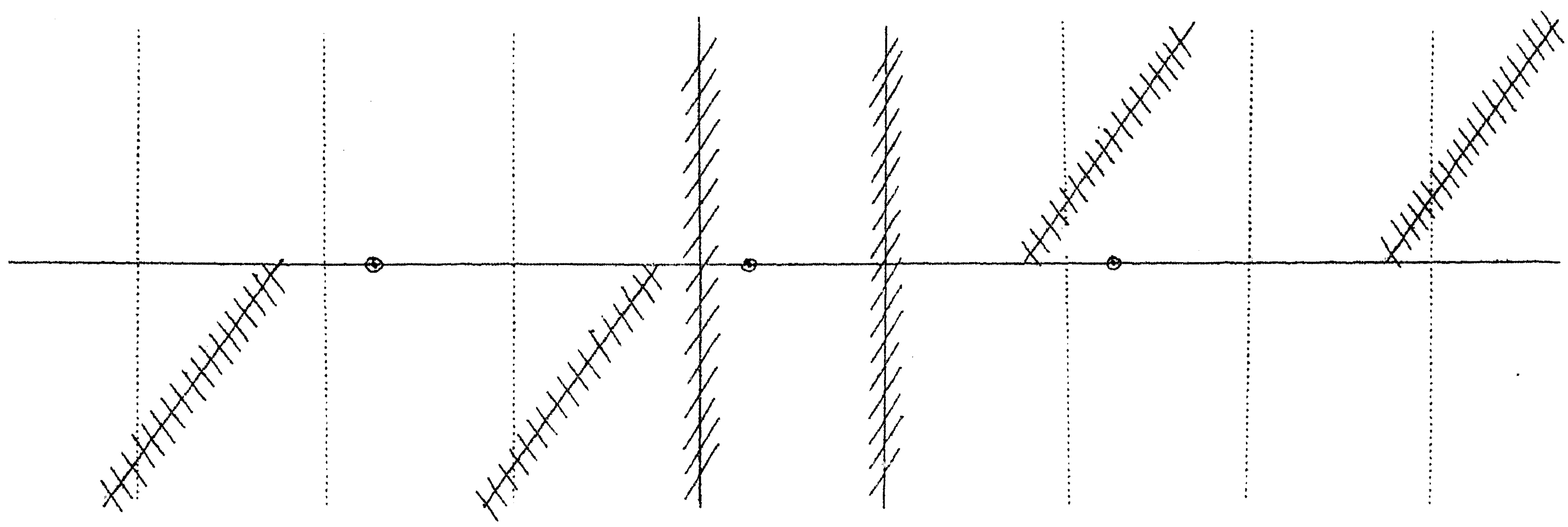
$$w = K_0(r) + \int_{-\infty}^{\infty} \frac{1}{4\,\mathrm{sh}(2a\,\mathrm{ch}v)} \left\{ e^{-(2a+\xi-x)\mathrm{ch}v} + e^{-(2a+x-\xi)\mathrm{ch}v} + \frac{\mathrm{sh}(v+\omega i)}{\mathrm{sh}(v-\omega i)} e^{(x+\xi)\mathrm{ch}v} + \frac{\mathrm{sh}(v-\omega i)}{\mathrm{sh}(v+\omega i)} e^{-(x+\xi)\mathrm{sh}v} \right\} e^{-(y-\eta)i\,\mathrm{sh}v}\, dv \tag{6.5}$$

In view of the expansion

$$\frac{1}{2\,\mathrm{sh}(2a\,\mathrm{ch}v)} = \sum_{0}^{\infty} e^{-(4n+2)a\,\mathrm{ch}v},$$

the expression 6.5 can be interpreted as the sum of contributions from the singularities obtained from (ξ, η) by repeated reflection with respect to the boundaries $x = \pm a$.

We get simple poles at the points $(\xi \pm 4na, \eta)$ and dipole tails radiating from the points $(2a-\xi \pm 4na, \eta)$ as indicated below.



Next we consider a similar problem in which a coast condition and an ocean condition are combined.

To determine a function $w(x,y,\xi,\eta)$ which satisfies

$$0 < y < a \qquad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = w \,, \qquad 6.6$$

$$y = 0 \qquad \frac{\partial w}{\partial x} \cos\omega + \frac{\partial w}{\partial y} \sin\omega = 0 \,, \qquad 6.7$$

$$y = a \qquad w = 0 \,, \qquad 6.8$$

and $w \sim -\ln r$ in (ξ, η).

We write the solution in the following form

$$w = K_0 \left\{ \sqrt{(x-\xi)^2+(y-\eta)^2} \right\} - K_0 \left\{ \sqrt{(x-\xi)^2+(y-2a+\eta)^2} \right\} +$$

$$+ 2 \int_{-\infty}^{\infty} \psi(v)\ \mathrm{sh}\left\{ (y-a)\mathrm{ch}v \right\}\ \mathrm{sh}\left\{ (\eta-a)\mathrm{ch}v \right\}\ e^{-(x-\xi)i\mathrm{sh}v}\, dv, \qquad 6.9$$

so that the condition 6.8 already has been satisfied.

Again we have

$$K_0 \left\{ \sqrt{(x-\xi)^2 + (y-\eta)^2} \right\} = \tfrac{1}{2} \int_{-\infty}^{\infty} e^{-|\eta-y|\,\mathrm{ch}v \,-\, (x-\xi)i\mathrm{sh}v}\, dv,$$

$$K_0 \left\{ \sqrt{(x-\xi)^2 + (y-2a+\eta)^2} \right\} = \tfrac{1}{2} \int_{-\infty}^{\infty} e^{-(2a-y-\eta)\mathrm{ch}v-(x-\xi)i\mathrm{sh}v}\, dv.$$

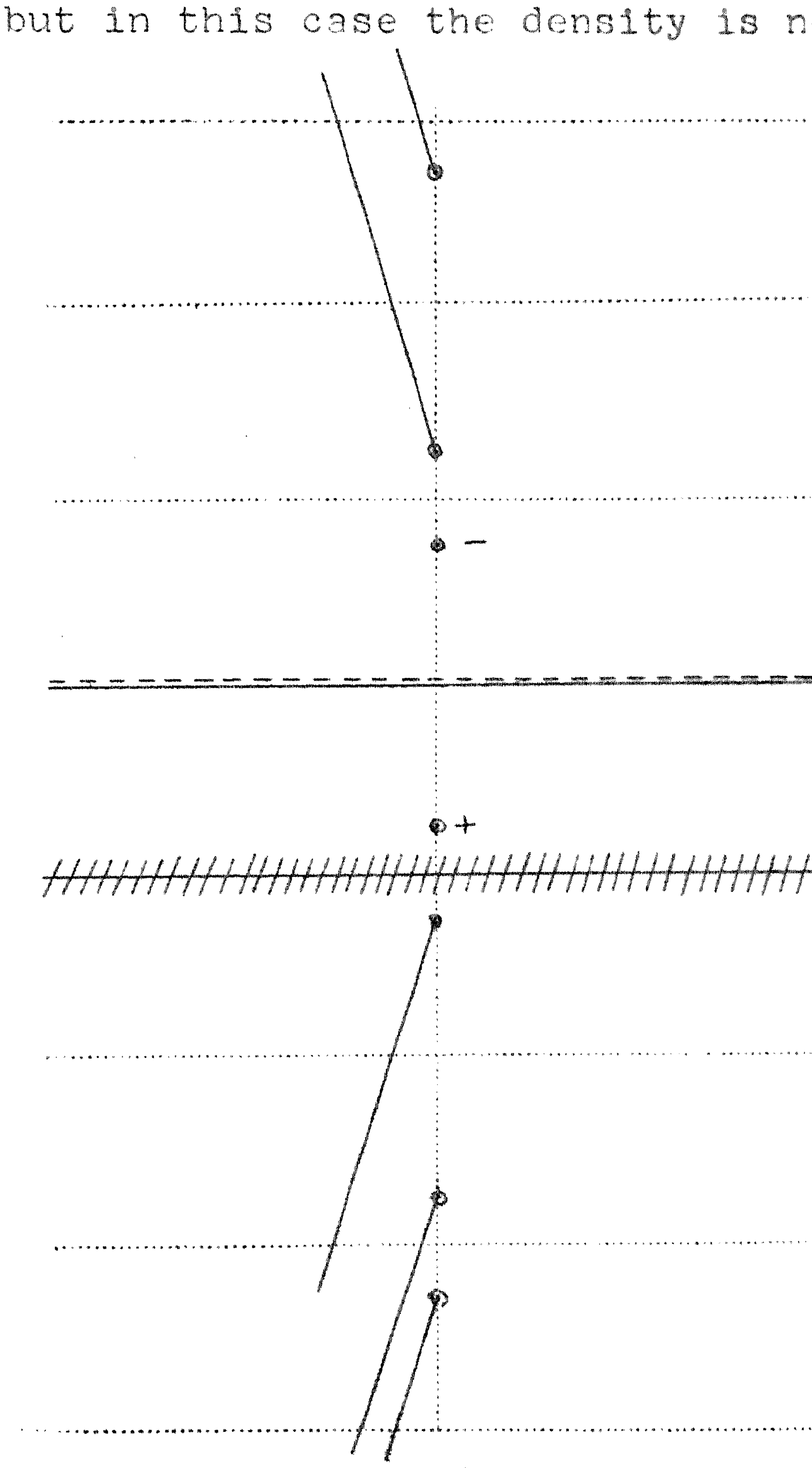
The boundary condition 6.7 gives

$$(-\mathrm{chv}\ \sin\omega + i\,\mathrm{shv}\ \cos\omega)\ e^{-a\,\mathrm{chv}} + 2\psi\{\mathrm{chv}\ \sin\omega\ \mathrm{ch}(a\,\mathrm{chv}) + i\,\mathrm{shv}\ \cos\omega\ \mathrm{sh}(a\,\mathrm{chv})\} = 0,$$

so that

$$\psi(v) = \frac{1}{1+\frac{\mathrm{sh}(v-\omega i)}{\mathrm{sh}(v+\omega i)}\ \exp\ (2a\,\mathrm{chv})} . \qquad 6.10$$

The series expansion of 6.10 resolves 6.9 into a sum of contributions from singularities obtained by repeated reflexions, skew and normal, from (ξ,η). Again the singularities can be considered as poles and lines of dipoles but in this case the density is no longer constant along each line.



§7. Rectangle.

According to the third method of §3 the following problem will be considered.

To determine a function $w(x,y,\xi,\eta)$ which satisfies the equation of Laplace

$$\Delta w = 0 , \qquad 7.1$$

which has a logarithmic singularity in (ξ, η) of the following kind

$$w \sim -\ln r \qquad r = \sqrt{(x-\xi)^2+(y-\eta)^2} \text{ for } r \to 0,$$

and which satisfies the boundary conditions

$$\text{at } C_1 \qquad \frac{\partial w}{\partial n} \sin\omega - \frac{\partial w}{\partial s} \cos\omega = 0 , \qquad 7.2$$

$$\text{at } C_2 \qquad w = 0 . \qquad 7.3$$

where ω is determined by $\cot g\, \omega = \frac{\Omega}{p+\lambda}$. The angle ω is not necessarily real since p may take complex values. If, however, we restrict ourselves to the positive real axis of the complex p-plane ω can be considered as a real angle in the interval $(0, \pi/2)$. By means of analytic continuation the value of G can be determined for complex p.

The function $w(x,y,\xi,\eta)$ can be considered as the real part of an analytic function of $z = x+yi$. We shall also temporarily write $\zeta = \xi+\eta i$. If we put

$$w = \mathrm{Re} \left\{ L(z) \right\} , \qquad 7.4$$

the conditions 7.2 and 7.3 can be written as

$$\text{at } C_1 \qquad \mathrm{Re} \left\{ e^{i(\omega+\tau)} \frac{dL}{dz} \right\} = 0, \qquad 7.5$$

$$\text{at } C_2 \qquad \mathrm{Re}\ L = 0. \qquad 7.6$$

where τ is defined as in § 2.

By means of the principle of conformal mapping this problem can be reduced to the much simpler problem of finding a solution for the half-plane $y > 0$ where C_1 and C_2 are respectively the negative and the positive part of the real axis.

Thus we consider the problem of finding an analytic function $L_o(z)$ such that for

$$y = 0 \quad x > 0 \qquad \mathrm{Re}\ (e^{i\omega} \frac{dL_o}{dz}) = 0 , \qquad 7.71$$

$$y = 0 \quad x < 0 \qquad \mathrm{Re}\ L_o = 0 , \qquad 7.72$$

and which has a logarithmic singularity at $z = \zeta$.

The derivative L' of $L_o(z)$ satisfies the conditions

$$y = 0 \quad x > 0 \qquad \mathrm{Re}\ (e^{i\omega} L_o') = 0 , \qquad 7.81$$

$$y = 0 \quad x < 0 \qquad \mathrm{Re}\ L_o' = 0 , \qquad 7.82$$

and for $z \to \zeta$ $\qquad L_o' \sim -\frac{1}{z-\zeta}$.

Obviously any function $z^{-\frac{\omega}{\pi}}\psi(z)$ where $\psi(z)$ is real on the real axis satisfies the conditions 7.8. For $z \to \zeta$ we must have

$$\psi(z) \sim - \frac{\zeta^{\frac{\omega}{\pi}}}{z-\zeta} ,$$

so that we may take

$$\psi(z) = - \frac{\zeta^{\frac{\omega}{\pi}}}{z-\zeta} + \frac{\zeta^{*\frac{\omega}{\pi}}}{z-\zeta^*} .$$

Finally L_o becomes

$$L_o(z) = \int_z^{\infty} s^{-\frac{\omega}{\pi}} \left(\frac{\zeta^{\frac{\omega}{\pi}}}{s-\zeta} + \frac{\zeta^{*\frac{\omega}{\pi}}}{s-\zeta^*} \right) ds. \qquad 7.9$$

From 7.9 the solution for the rectangle can be obtained by means of conformal representation. Let $M(z)$ be the mapping function which maps the rectangle upon the upper halfplane then we have

$$z \longrightarrow M(z)$$
$$\zeta \longrightarrow M(\zeta)$$
$$L_o(z) \longrightarrow L(z)$$

with

$$L(z) = \int_{M(z)}^{\infty} s^{-\frac{\omega}{\pi}} \left\{ \frac{\{M(\zeta)\}^{\frac{\omega}{\pi}}}{s-M(\zeta)} - \frac{\{M^*(\zeta)\}^{\frac{\omega}{\pi}}}{s-M^*(\zeta)} \right\} ds. \qquad 7.10$$

If now the incomplete Beta-function $H(z, \lambda)$ is defined by

$$H(z, \lambda) = \int_z^{\infty} \frac{t^{-\lambda}}{t-1}\, dt \qquad 0 < \lambda < 1 \qquad 7.11$$

the function $L(z)$ of 7.10 may be written as

$$L(z) = H\left\{ \frac{M(z)}{M(\zeta)}, \frac{\omega}{\pi} \right\} - H\left\{ \frac{M(z)}{M^*(\zeta)}, \frac{\omega}{\pi} \right\} . \qquad 7.12$$

The function $H(z, \lambda)$ can be expanded in a power series of $\frac{1}{z}$

$$H(z, \lambda) = \frac{z^{-\lambda}}{\lambda} \sum_0^{\infty} \frac{z^{-j}}{\lambda+j} , \qquad |z| > 1 , \qquad 7.13$$

or

$$H(z, \lambda) = \frac{z^{-\lambda}}{\lambda} F(1, \lambda, \lambda+1, z^{-1}), \qquad 7.14$$

where F is the hypergeometric function.

The transformations of the hypergeometric function may lead to alternative expansions e.g.

$$H(z, \lambda) = C + \frac{z^{1-\lambda}}{1-\lambda} F(1, 1-\lambda, 2-\lambda, z), \quad |z| < 1, \qquad 7.15$$

$$H(z,\lambda) = C - \ln(1-z) + z^{1-\lambda} \sum_{0}^{\infty} \frac{(1-\lambda)_n}{n!} \left\{ \psi(n+1) - \psi(n+1-\lambda) \right\} (1-z)^n,$$

$$|1-z| < 1, \qquad 7.16$$

where $\psi(t) = \frac{\Gamma'(t)}{\Gamma(t)}$ and $C = \frac{\pi}{\sin \lambda\pi} e^{\pm \lambda \pi i}$.

The mapping function $M(z)$ for a ractangular region can be derived from the Weierstrassian elliptic function $\wp(z)$.

The elliptic function $\wp(z)$ has the periods $2\omega_1$ and $2\omega_2$. It is customary to write $\omega_3 = -\omega_1 - \omega_2$ and $\wp(\omega_j) = e_j$ $(j=1,2,3)$.

If ω_1 is real and ω_2 is a pure imaginary the e_j are real, $e_1 > e_3 > e_2$, and $s = \wp(z)$ satisfies the differential equation

$$\wp'(z) = 2 \left\{ (s-e_1)(s-e_2)(s-e_3) \right\}^{\frac{1}{2}}. \qquad 7.17$$

Consider now a rectangle with vertices at $-\frac{1}{2}\omega_1$, $\frac{1}{2}\omega_1$, $\omega_2 + \frac{1}{2}\omega_1$, $\omega_2 - \frac{1}{2}\omega_1$. Then the function

$$M(z) = \wp(z - \tfrac{1}{2}\omega_1 - \omega_2) - e_1 \qquad 7.18$$

takes real values on the sides of this rectangle and maps the interior of the rectangle on the upper halfplane. In particular we have

$$M(-\tfrac{1}{2}\omega_1) = e_3 - e_1 < 0.$$

$$M(\tfrac{1}{2}\omega_1) = e_2 - e_1 < 0.$$

$$M(\omega_2 + \tfrac{1}{2}\omega_1) = \infty \qquad 7.19$$

$$M(\omega_2 - \tfrac{1}{2}\omega_1) = 0.$$

In our case the periods $2\omega_1$ and $2\omega_2$ are 4a and 2bi respectively and the period-parallelogram turns out to be twice as large as the original rectangle.

At this stage we have reached the following result

If $\zeta(x,y,t)$ is the elevation of the surface of the rectangular model R of the North Sea due to a windfield (W_x, W_y) then the Laplace transform $\bar{\zeta}$ satisfies the following integral equation

$$\bar{\zeta}(\xi,\eta,p) = k^2 \iint_R \bar{G}(x,y,\xi,\eta,p)\,\bar{\zeta}(x,y,p)\,dx\,dy \; + $$

$$+ \iint_R \bar{G}(x,y,\xi,\eta,p)\,\bar{F}(x,y,p)\,dx\,dy$$

$$- \int_{C_1} \bar{G}(s,\xi,\eta,p)\,\bar{\Gamma}(s,p)\,ds \; ; \qquad 7.20$$

where

$$k^2 = p\,\frac{(p+\lambda)^2 + \Omega^2}{gh\,(p+\lambda)} \;,$$

$$\bar{F} = \frac{1}{gh\varrho}\left\{ \left(\frac{\partial \bar{W}_x}{\partial x} + \frac{\partial \bar{W}_y}{\partial y}\right) + \frac{\Omega}{p+\lambda}\left(\frac{\partial \bar{W}_y}{\partial x} - \frac{\partial \bar{W}_x}{\partial y}\right)\right\} \;,$$

$$\bar{\Gamma} = \frac{1}{gh\varrho}\left(\bar{W}_n + \frac{\Omega}{p+\lambda}\,\bar{W}_s\right) \;,$$

$$\bar{G}(x,y,\xi,\eta,p) = \frac{1}{4\pi}\,H\left\{\frac{M(x+yi)}{M(\xi+\eta i)}\right\} + \frac{1}{4\pi}\,H\left\{\frac{M(x-yi)}{M(\xi-\eta i)}\right\} -$$

$$- \frac{1}{4\pi}\,H\left\{\frac{M(x-yi)}{M(\xi+\eta i)}\right\} - \frac{1}{4\pi}\,H\left\{\frac{M(x+yi)}{M(\xi-\eta i)}\right\} \;;$$

and $$H(z) = \int_z^{\infty} \frac{t^{-\frac{\omega}{\pi}}}{t-1}\,dt \;,$$

with $$\operatorname{cotg}\omega = \frac{\Omega}{p+\lambda} \;.$$

and $$M(z) = \wp(z-2a-2bi) - \wp(2a)$$

where $\wp(z)$ is the double-periodic Weierstrass function with periods $4a$ and $2bi$ respectively.

The integral equation 7.20 might be solved by means of iteration so that for $\bar{\zeta}$ a Neumann-series is obtained. It has been said before that the convergency of this series can be improved if the lower characteristic values of the proper oscillation are known.

If $\bar{\zeta}$ (x,y,p) is known for sufficiently many values of p,which in itself involves a considerable amount of labour, the backward transformation from p to time t should be performed. This process offers less difficulties since it is a standard procedure in the practical application of the Laplace transform where a number of approximate methods are known.

Thus, in principle, the problem stated at the beginning of this report, has been solved.